N14/5/MATHL/HP3/ENG/TZ0/SE

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MATHEMATICS
HIGHER LEVEL
PAPER 3 – CALCULUS

Thursday 13 November 2014 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics HL*** *formula booklet* is required for this paper.
- The maximum mark for this examination paper is *[60 marks]*.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 14]*

(a) Use the integral test to determine the convergence or divergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}.$$ *[3]*

(b) Let $S = \sum_{n=1}^{\infty} \frac{(x+1)^n}{2^n \times n^{0.5}}$.

(i) Use the ratio test to show that S is convergent for $-3 < x < 1$.

(ii) Hence find the interval of convergence for S. *[11]*

2. *[Maximum mark: 14]*

(a) Use an integrating factor to show that the general solution for $\frac{dx}{dt} - \frac{x}{t} = -\frac{2}{t}, t > 0$ is $x = 2 + ct$, where c is a constant. *[4]*

The weight in kilograms of a dog, t weeks after being bought from a pet shop, can be modelled by the following function:

$$w(t) = \begin{cases} 2 + ct & 0 \le t \le 5 \\ 16 - \frac{35}{t} & t > 5 \end{cases}.$$

(b) Given that $w(t)$ is continuous, find the value of c. *[2]*

(c) Write down

(i) the weight of the dog when bought from the pet shop;

(ii) an upper bound for the weight of the dog. *[2]*

(d) Prove from first principles that $w(t)$ is differentiable at $t = 5$. *[6]*

3. *[Maximum mark: 10]*

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ where $f(x, y) = y - 2x$.

(a) Sketch, on one diagram, the four isoclines corresponding to $f(x, y) = k$ where k takes the values $-1, -0.5, 0$ and 1. Indicate clearly where each isocline crosses the y axis. *[2]*

A curve, C, passes through the point $(0, 1)$ and satisfies the differential equation above.

(b) Sketch C on your diagram. *[3]*

(c) State a particular relationship between the isocline $f(x, y) = -0.5$ and the curve C, at their point of intersection. *[1]*

(d) Use Euler's method with a step interval of 0.1 to find an approximate value for y on C, when $x = 0.5$. *[4]*

4. *[Maximum mark: 22]*

In this question you may assume that $\arctan x$ is continuous and differentiable for $x \in \mathbb{R}$.

(a) Consider the infinite geometric series

$$1 - x^2 + x^4 - x^6 + \ldots \qquad |x| < 1.$$

Show that the sum of the series is $\frac{1}{1+x^2}$. *[1]*

(b) Hence show that an expansion of $\arctan x$ is $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$ *[4]*

(c) f is a continuous function defined on $[a, b]$ and differentiable on $]a, b[$ with $f'(x) > 0$ on $]a, b[$.

Use the mean value theorem to prove that for any $x, y \in [a, b]$, if $y > x$ then $f(y) > f(x)$. *[4]*

(d) (i) Given $g(x) = x - \arctan x$, prove that $g'(x) > 0$, for $x > 0$.

(ii) Use the result from part (c) to prove that $\arctan x < x$, for $x > 0$. *[4]*

(e) Use the result from part (c) to prove that $\arctan x > x - \frac{x^3}{3}$, for $x > 0$. *[5]*

(f) Hence show that $\frac{16}{3\sqrt{3}} < \pi < \frac{6}{\sqrt{3}}$. *[4]*